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DETERMINATION OF THE PARAMETERS OF THE PLANAR FLOW OF AN INCOMPRESSIBLE FLUID WHEN THERE IS A SMALL VARIATION IN THE CONTOUR OF THE PROFILE†

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A formula is obtained in a corrected form for calculating the field of flow of an incompressible fluid on a profile close to a specified profile.

SUPPOSE the flow around a certain profile C is known. Then, a convenient formula [1], which is given in all editions of the monograph [2, p. 395] has been obtained for calculating the flow on a profile C_1 which is close to it (Fig. 1). One correction in the derivation of the above-mentioned formula has been pointed out in [3]. There are, however, still errors in the derivation. Below, we present an additional correction to the formula for the velocity distribution over a contour which is close to a specified profile.

Let us specify [2] a conformal mapping of the exterior of the profile C onto the exterior of a unit circle by the function

$$\zeta = F(z,C), \quad F(\infty,C)=\infty \tag{1}$$

The function (1) defines the correspondence of points of C and points of the circle $\zeta = e^{i\theta}$ $(s=s(\theta)$, where s is the length of an arc along the profile C). We will denote by $n(s)$ the length of a segment of the external normal $\mathbf{n}$ to the contour C. In the case of the mapping (1), the line C passes into the line C_1^*, the equation of which, up to terms of the second order of smallness with respect to $|n(s)|$, in logarithmic coordinates has the form

$$\rho = 1 + n[s(\theta)]d\theta/ds = 1+\delta(\theta) \tag{2}$$

We will assume that the deviation $\delta=\delta(\theta)$ of the curve C_1^* from a circle of unit radius is small such that $|\delta|<\varepsilon$, $|\delta'|<\varepsilon$ and $\delta''|<\varepsilon$, where ε is a small quantity.

The mapping of the exterior of C_1 onto the exterior of the unit circle $|w|<1$ can be represented by the superposition

$$w = F_1(z,C_1) = F_2[F(z,C_1),C_1^*] \tag{3}$$

where $w=F_2(\zeta,\ C_1^*)$ is the mapping of the exterior of C_1^* onto $|w|>1$. On differentiating formula (3), we obtain

$$|F_1'(z,C_1)| = |F_2'(\zeta,C_1^*)||F_1'(z,C_1)| \tag{4}$$

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The first factor on the right-hand side is determined from the theory of conformal mappings of close domains [2] and, when account is taken of the correction in [3], it has the form

$$|F_2'(\zeta,C_1^*)|\cong 1-\delta(\theta)-\gamma(\theta),\qquad \gamma(\theta)=\frac{1}{4\pi}\int\limits_0^{2\pi}[\delta(t)-\delta(\theta)]\sin^{-2}\frac{t-\theta}{2}dt \tag{5}$$

The second factor on the right-hand side of (4) can be determined, by expanding the function $F'(z, C_1)$ in a Taylor series in the neighbourhood of the point z of contour C up to terms of the second order of smallness

$$F'(z,C_1)=F'(z,C)+F''(z,C)\Delta z+... \tag{6}$$

where Δz is the distance between points on the contours C and C_1 lying on the normal to the contours C.

The mapping (3) reduces the problem of the flow around the contour C_1 to the problem of the flow around a circular cylinder, and the magnitude of the velocity on C is therefore given by the formula

$$|\upsilon_1|=2\upsilon_\infty|\sin\vartheta-\sin\vartheta_0||F_1'(z,C_1)|/|F_1'(\infty,C_1)| \tag{7}$$

Here υ_∞ is the value of the velocity at infinity, directed along the real axis and $\vartheta=\theta+\Delta\theta$, $\vartheta_0=\theta_0+\Delta\theta_0$ are the arguments of the images of the points z and z_0 (a fixed point) in the case of the mapping (3).

Allowing for the fact that the velocity on the contour C

$$|\upsilon|=2\upsilon_\infty|\sin\theta-\sin\theta_0||F'(z,C)|/|F'(\infty,C)|$$

and making use of formulae (4)–(6) and again (4) for $z=\infty$, we obtain

$$|\upsilon_1|=\frac{|\upsilon|}{|F_2'(\infty,C_1^*)|}\left|\frac{\sin\vartheta-\sin\vartheta_0}{\sin\theta-\sin\theta_0}\right|\left|1+\frac{F''(z,C)}{F'(z,C)}\Delta z\right|(1-\delta(\theta)-\gamma(\theta)) \tag{8}$$

where

$$|F_2'(\infty,C_1^*)|\approx 1-\frac{1}{2\pi}\int\limits_0^{2\pi}\delta(t)dt$$

$$\frac{\sin\vartheta-\sin\vartheta_0}{\sin\theta-\sin\theta_0}\approx 1+\frac{\cos\theta\Delta\theta-\cos\theta_0\Delta\theta_0}{\sin\theta-\sin\theta_0}$$

$$\Delta\theta=\frac{1}{2\pi}\int\limits_0^{2\pi}\delta(t)\operatorname{ctg}\frac{\theta-t}{2}dt,\quad \Delta\theta_0=\frac{1}{2\pi}\int\limits_0^{2\pi}\delta(t)\operatorname{ctg}\frac{\theta_0-t}{2}dt$$

As a result, we obtain the required relationship which connects the velocities at the corresponding points A and A_1 (Fig. 1) of the contours C and C_1 which lie on a single normal to C

$$\begin{aligned}|\upsilon_1|=|\upsilon|\Gamma\Bigg(&1-\delta(\theta)+\frac{\cos\theta\Delta\theta-\cos\theta_0\Delta\theta_0}{\sin\theta-\sin\theta_0}+\frac{1}{2\pi}\int\limits_0^{2\pi}\delta(t)dt-\\ &-\frac{1}{4\pi}\int\limits_0^{2\pi}[\delta(t)-\delta(\theta)]\sin^{-2}\frac{t-\theta}{2}dt\Bigg),\quad \Gamma=|1+\Delta zF''(z,C)/F'(z,C)|\end{aligned} \tag{9}$$

Expression (9) is the corrected version of the well-known formula in [2] which differs by the presence of the factor Γ and the sign in front of the second term in the parentheses. Note that, according to (9), the critical points K and K_1 (Fig. 1) lie on a single normal to the contour C.

Let us check the result which has been obtained. As the reference profile, we will consider an ellipse

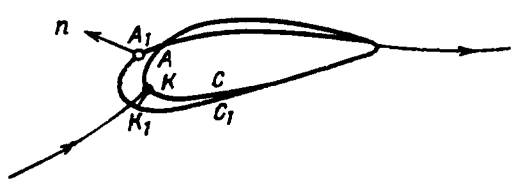

FIG. 1.

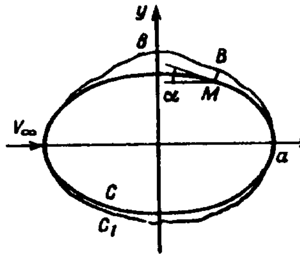

FIG. 2.

(Fig. 2) specified in parametric form $\{x = a\cos\varphi,\ y = b\sin\varphi\}$ and we will define a close, arbitrary contour by the equation

$$x = a\cos\varphi + \varepsilon m\Phi(\varphi),\quad y = \varepsilon\sin\varphi + \varepsilon p\Phi(\varphi) \tag{10}$$

where ε is a small quantity, which characterizes the closeness of the contours C and C_1 and $\{m, p\} = \{2\cos\varphi / a,\ 2\sin\varphi / b\}$ are the components of the vector of the normal to the ellipse at an arbitrary point $\Phi(\varphi)$ which determines the variation of the contour. Without dwelling on the details, which can be found in [4] (expression (12)), we obtain a formula in which a factor Γ (9) of the form

$$\Gamma = 1 + \mu\delta(\varphi),\quad \mu = 1 - \frac{(1+\mathrm{ctg}^2\varphi)b/a}{1+\mathrm{ctg}^2\varphi b^2/a^2} \tag{11}$$

will occur.

In this case, formula (9), when account is taken of the equalities $\theta = \varphi$ and $\theta_0 = 0$ yields

$$\upsilon_s = |\upsilon_1| = \left(1+\frac{b}{a}\right)|\cos\alpha_1|\Bigg\{1+\varepsilon\Bigg[q\sin^2\alpha_1 - \frac{2}{ab}\Phi(\varphi)(1-\mu) +$$

$$+\frac{1}{\pi ab\sin\varphi}\int_0^{2\pi}\Phi(t)\mathrm{ctg}\frac{t}{2}dt + \frac{\mathrm{ctg}\varphi}{\pi ab}\int_0^{2\pi}\Phi(t)\mathrm{ctg}\frac{\varphi - t}{2}dt +$$

$$+\frac{1}{\pi ab}\int_0^{2\pi}\Phi(t)dt - \frac{1}{2\pi ab}\int_0^{2\pi}[\Phi(t)-\Phi(\varphi)]\sin^{-2}\frac{t-\varphi}{2}dt\Bigg]\Bigg\}$$

$$q = 2\left[\frac{\cos\varphi\Phi(\varphi)+\sin\varphi\Phi'_\varphi(\varphi)}{b^2\cos\varphi} + \frac{\cos\Phi'_\varphi(\varphi) - \sin\varphi\Phi(\varphi)}{a^2\sin\varphi}\right]$$

where α_1 is the local angle of attack to the contour C_1 which is expressed in terms of the local angle of at tack to the ellipse $\mathrm{tg}\alpha_1 = \mathrm{tg}\alpha(1+\varepsilon q)$.

Let us now fix the close contour by specifying its equation in the plane ζ in the form of a circle: $\rho = 1 + \delta(\theta) = 1 + a_0$, $\zeta = \rho e^{i\theta}$. It can be shown that, in this case, the contour C_1 is an ellipse with semi-axes a_1 and b_1, the equation of which in parametric form is: $x = a_1\cos\varphi = (a + a_0 b)\cos\varphi$, $y = b_1\sin\varphi = b_1\sin\varphi = (b + a_0 a)\sin\varphi$. Using (12), we obtain $\upsilon_s = (1 + b/a)[1 + a_0(1 - b/a)]\cos\alpha_1$.

On the other hand, the exact velocity distribution over an ellipse [4] is

$$\upsilon_s = (1 + b_1/a_1)\cos\alpha_1,\quad a_1 = a + a_0 b,\quad b_1 = b + a_0 a(a_0 \sim \varepsilon)$$

Since $b_1/a_1 = (b/a) + (1 - b^2/a^2)a_0 + \ldots$ the results are identical.

Note that, if we had taken a circle as the reference contour in the test example, the factor $\Gamma \equiv 1$ and its absence cannot be observed [4].

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Translated E.L.S.